Higher dimensional bright solitons and their collisions in a multicomponent long wave–short wave system

2009 J. Phys. A: Math. Theor. 42 115103

(http://iopscience.iop.org/1751-8121/42/11/115103)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 115103 (20pp) doi:10.1088/1751-8113/42/11/115103

Higher dimensional bright solitons and their collisions in a multicomponent long wave–short wave system

T Kanna[1], **M Vijayajayanthi**[2], **K Sakkaravarthi**[1] **and M Lakshmanan**[2]

[1] Department of Physics, Bishop Heber College, Tiruchirapalli-620017, India
[2] Centre for Nonlinear Dynamics, School of Physics, Bharathidasan University, Tiruchirapalli-620024, India

E-mail: lakshman@cnld.bdu.ac.in

Received 14 August 2008, in final form 28 January 2009
Published 23 February 2009
Online at stacks.iop.org/JPhysA/42/115103

Abstract

Bright plane soliton solutions of an integrable (2+1)-dimensional $(n + 1)$-wave system are obtained by applying Hirota's bilinearization method. First, the soliton solutions of a three-wave system consisting of two short-wave components and one long-wave component are found and then the results are generalized to the corresponding integrable $(n + 1)$-wave system with n short waves and a single long wave. It is shown that the solitons in the short-wave components (say $S^{(1)}$ and $S^{(2)}$) can be amplified by merely reducing the pulse width of the long-wave component (say L). Study of the collision dynamics reveals some interesting behaviour: the solitons which split up in the short-wave components undergo shape changing collisions with intensity redistribution and amplitude-dependent phase shifts. Even though a similar type of collision is possible in (1+1)-dimensional multicomponent integrable systems, to our knowledge we report this kind of collision in (2+1) dimensions for the first time. However, solitons which appear in the long-wave component exhibit only elastic collision though they undergo amplitude-dependent phase shifts.

PACS numbers: 02.30.Jr, 05.45.Yv

1. Introduction

One of the main emphases of current research in the area of integrable systems and their applications is the study of multicomponent nonlinear systems admitting soliton-type solutions [1–17]. In (1+1) dimensions, it has been shown that the multicomponent bright solitons of the integrable N-coupled nonlinear Schrödinger (CNLS) equations undergo fascinating shape changing collisions with intensity redistribution which have no single component counterpart [4–8]. This interesting behaviour has found applications in nonlinear switching devices [12], matter wave switches [13] and more importantly in the context of optical computing in bulk media [14, 15]. There is a natural tendency to look for such a collision type in higher dimensions. From this point of view, we have considered the following recently studied integrable coupled (2+1)-dimensional ((2+1)D) system, which is a two-component analogue of the two-dimensional long wave–short wave resonance interaction (LSRI) system [16], in a dimensionless form,

$$\mathrm{i}\left(S_t^{(j)} + S_y^{(j)}\right) - S_{xx}^{(j)} + LS^{(j)} = 0, \qquad j = 1, 2, \tag{1a}$$

$$L_t = 2\sum_{j=1}^{2} |S^{(j)}|_x^2, \tag{1b}$$

where the subscripts denote partial derivatives. (Note that in the above two components mean two short-wave (S) components.) The one-component ($j = 1$) version of the above equations corresponds to the interaction of a long interfacial wave (L) and a short surface wave (S) in a two-layer fluid [18]. Also in [19], the existence of a dromion-like solution was established for the $j = 1$ case. In their very recent interesting work, Ohta, Maruno and Oikawa [16] derived equations (1) as the governing equations for the interaction of three nonlinear dispersive waves by applying a reductive perturbation method. Here, among these three waves, two waves propagate in the anomalous dispersion region and the third wave propagates in the normal dispersion regime. In the context of long wave–short wave interaction, the first two components can be viewed as the two components of the short surface waves while the last component corresponds to the long interfacial wave. Note that the presence of the long interfacial wave induces nonlinear interaction between the two short-wave components which leads to nontrivial collision behaviour as will be shown in this paper. From here onwards we refer to equation (1) as the three-wave LSRI system in which the first two components correspond to short waves and the last one is a long wave.

Apart from deriving the governing equation (1) in [16], Ohta *et al* also gave Wronskian-type soliton solutions of a specific type where the components $S^{(1)}$, $S^{(2)}$ and L consist of N solitons, M solitons and $(M + N)$ solitons, respectively. In this context, however, it is of considerable interest to study the collision behaviour if the same number of solitons is split up in all the three components, to check whether nontrivial shape changing collisions of solitons as in the case of CNLS systems [4, 5] also occur here and to look for the possibilities of construction of logic gates based on soliton collisions. For the one-component case the interaction of two solitons in both short-wave and long-wave components has been studied in detail in [18] and certain interesting features such as fusion and fission processes have been revealed. In this study, we consider the multicomponent (2+1)D LSRI system, admitting the same number of bright solitons in all the three components, and obtain multisoliton solutions. Our analysis of their collision properties shows that the solitons appearing in the short-wave components exhibit a shape changing collision scenario resulting in a redistribution of intensity as well as amplitude-dependent phase shift, whereas the long-wave component solitons undergo standard elastic collisions only, but with amplitude-dependent phase shifts. We also point out that the $(N, M, N + M)$-soliton solutions obtained in [16] follow as special cases of the (m, m, m)-multisoliton solution obtained here when some of the soliton parameters are restricted to very special values. The study is also extended to the $(n + 1)$-wave system as well, where n is arbitrary.

The plan of the paper is as follows: in section 2, we briefly present the bilinearization procedure for the three-wave system. The multisoliton solution of the three-wave system is discussed in section 3. Explicit one-soliton and two-soliton solutions are analysed in section 4.

The asymptotic analysis of the two-soliton solution of the three-wave system is given in section 5. The interesting collision scenario of two solitons is discussed in detail in section 6. Sections 7 and 8 deal with three- and four-soliton solutions, respectively. The multicomponent case with $j > 2$ in equation (1) is studied in section 9. Section 10 is allotted to the conclusion.

2. (2+1)D bright soliton solutions

The soliton solutions of equation (1) are obtained by using Hirota's direct method [20, 21]. By performing the bilinearizing transformations,

$$S^{(j)} = \frac{g^{(j)}}{f}, \qquad L = -2\frac{\partial^2}{\partial x^2}(\log f), \qquad j = 1, 2, \tag{2}$$

where $g^{(j)}$ are complex functions while f is a real function, equation (1) can be decoupled into the following bilinear equations:

$$\left(\mathrm{i}(D_t + D_y) - D_x^2\right)(g^{(j)} \cdot f) = 0, \qquad j = 1, 2, \tag{3a}$$

$$D_t D_x (f \cdot f) = -2\sum_{j=1}^{2}(g^{(j)} g^{(j)*}), \tag{3b}$$

where $*$ denotes the complex conjugate. Hirota's bilinear operators D_x, D_y and D_t are defined as

$$D_x^p D_y^q D_t^r (a \cdot b) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^p \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^q \times \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^r a(x, y, t) b(x', y', t')\Big|_{(x=x', y=y', t=t')}.$$

Expanding $g^{(j)}$ and f formally as power series expansions in terms of a small arbitrary real parameter χ,

$$g^{(j)} = \chi g_1^{(j)} + \chi^3 g_3^{(j)} + \cdots, \qquad j = 1, 2, \tag{4a}$$

$$f = 1 + \chi^2 f_2 + \chi^4 f_4 + \cdots \tag{4b}$$

and solving the resultant set of linear partial differential equations recursively, one can obtain the explicit forms of $g^{(j)}$ and f. Then by substituting their expressions into (2) one can write the soliton solutions. The procedure has successfully been used to unearth several interesting properties of soliton collisions associated with the CNLS system in [4–8]. We have used a similar procedure here and obtained the one-soliton (1, 1, 1), two-soliton (2, 2, 2), three-soliton (3, 3, 3) and four-soliton (4, 4, 4) solutions explicitly. This can be generalized to the arbitrary m-soliton (m, m, m) solution in a Gram determinant form. From this one may claim that in the general case the number of solitons which split up in the short-wave components ($S^{(1)}$ and $S^{(2)}$) as well as in the long-wave component (L) is the same. However, we also point out that the (1, 1, 2)-, (2, 2, 4)- and $(N, M, N + M)$-soliton solutions obtained by Ohta *et al* [16] can be deduced as special cases of our (m, m, m)-soliton solution with $m = 2, m = 4$ and $m = N + M$, respectively, for particular choices of parameters in the solutions.

3. Arbitrary *m*-soliton solution

We first present the general form of (m, m, m)-soliton solution for an arbitrary m soliton in the following Gram determinant form. In order to write the multisoliton (m-soliton) solution of the three-wave LSRI system (1), we define the following $(1\times m)$ row matrix C_s, $s=1,2$, $(m\times 1)$ column matrices ψ_j and ϕ, $j=1,2,\ldots,m$ and the $(m\times m)$ identity matrix I:

$$C_s=-\left(\alpha_1^{(s)},\alpha_2^{(s)},\ldots,\alpha_m^{(s)}\right),\qquad \mathbf{0}=(0,0,\ldots,0),\tag{5a}$$

$$\psi_j=\begin{pmatrix}\alpha_j^{(1)}\\ \alpha_j^{(2)}\end{pmatrix},\qquad \phi=\begin{pmatrix}e^{\eta_1}\\ e^{\eta_2}\\ \vdots\\ e^{\eta_m}\end{pmatrix},\qquad I=\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}.\tag{5b}$$

Here, $\alpha_j^{(s)}$, $s=1,2$, $j=1,2,\ldots,m$ are arbitrary complex parameters and $\eta_i=k_ix-\left(ik_i^2+\omega_i\right)y+\omega_it$, $i=1,2,\ldots,m$ and k_i and ω_i are complex parameters. Then we can write the multisoliton solution of the three-wave LSRI system in the form of equation (2), with

$$S^{(s)}=\frac{g^{(s)}}{f},\qquad s=1,2,\qquad L=-2\frac{\partial^2}{\partial x^2}\log(f),\tag{6}$$

where

$$g^{(s)}=\begin{vmatrix}A&I&\phi\\-I&B&\mathbf{0}^T\\ \mathbf{0}&C_s&0\end{vmatrix},\qquad f=\begin{vmatrix}A&I\\-I&B\end{vmatrix},\tag{7a}$$

in which s denotes the short-wave components. Here the matrices A and B are defined as

$$A_{ij}=\frac{e^{\eta_i}+e^{\eta_j^*}}{k_i+k_j^*},\qquad B_{ij}=\kappa_{ji}=\frac{-\psi_i^\dagger\psi_j}{\left(\omega_i^*+\omega_j\right)}=-\frac{\left(\alpha_j^{(1)}\alpha_i^{(1)^*}+\alpha_j^{(2)}\alpha_i^{(2)^*}\right)}{\left(\omega_i^*+\omega_j\right)},$$
$$i,j=1,2,\ldots,m.\tag{7b}$$

In equation (7*b*), † represents the transpose conjugate and the real parts of ω_i (or k_i) should be chosen as negative quantities in order to obtain nonsingular solutions, which are necessary conditions. Sufficiency condition requires the choice of parameters such that f is real and nonzero (see sections 4, 7 and 8 for details in the case of $m=1,2,3$ and 4).

3.1. Proof of multisoliton solution of the three-wave LSRI system

We now prove that the Gram determinant forms of $g^{(s)}$ and f given above indeed satisfy the bilinear equations (3). By applying the derivative formula for the determinants, that is

$$\frac{\partial D}{\partial x}=\sum_{1\leqslant i,j\leqslant n}\frac{\partial a_{i,j}}{\partial x}\frac{\partial D}{\partial a_{i,j}}=\sum_{1\leqslant i,j\leqslant n}\frac{\partial a_{i,j}}{\partial x}\Delta_{i,j},\tag{8a}$$

where

$$D=\begin{vmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\ \vdots&\vdots&\ddots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{vmatrix}$$

and $\Delta_{i,j}$ is the cofactor of the (i, j)th element and making use of the properties of bordered determinants and also the elementary properties of determinants [21, 22], the derivatives $g_x^{(s)}, f_x, f_t, f_{xt}, g_z^{(s)}, g_{xx}^{(s)}, f_z$ and f_{xx}, where $\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)$, can be derived as below:

$$g_x^{(s)} = \begin{vmatrix} A & I & \phi & \phi_x \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & C_s & 0 & 0 \\ \mathbf{0} & \mathbf{0} & -1 & 0 \end{vmatrix}, \qquad f_x = \begin{vmatrix} A & I & \phi \\ -I & B & \mathbf{0}^T \\ -\phi^\dagger & \mathbf{0} & 0 \end{vmatrix}, \tag{8b}$$

$$f_t = -\sum_{s=1}^{2} \begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & -C_s^\dagger \\ \mathbf{0} & C_s & 0 \end{vmatrix}, \qquad f_{xt} = -\sum_{s=1}^{2} \begin{vmatrix} A & I & \phi & \mathbf{0}^T \\ -I & B & \mathbf{0}^T & -C_s^\dagger \\ -\phi^\dagger & \mathbf{0} & 0 & 0 \\ \mathbf{0} & C_s & 0 & 0 \end{vmatrix}, \tag{8c}$$

$$g_z^{(s)} = -\,\mathrm{i} \begin{vmatrix} A & I & \phi & \phi_{xx} \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & C_s & 0 & 0 \\ \mathbf{0} & \mathbf{0} & -1 & 0 \end{vmatrix} + \mathrm{i} \begin{vmatrix} A & I & \phi & \phi_x \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & C_s & 0 & 0 \\ \phi^\dagger & \mathbf{0} & 0 & 0 \end{vmatrix}, \tag{8d}$$

$$g_{xx}^{(s)} = \begin{vmatrix} A & I & \phi & \phi_{xx} \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & C_s & 0 & 0 \\ \mathbf{0} & \mathbf{0} & -1 & 0 \end{vmatrix} + \begin{vmatrix} A & I & \phi & \phi_x \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & C_s & 0 & 0 \\ \phi^\dagger & \mathbf{0} & 0 & 0 \end{vmatrix}, \tag{8e}$$

$$f_z = -\,\mathrm{i} \begin{vmatrix} A & I & \phi_x \\ -I & B & \mathbf{0}^T \\ -\phi^\dagger & \mathbf{0} & 0 \end{vmatrix} + \mathrm{i} \begin{vmatrix} A & I & \phi \\ -I & B & \mathbf{0}^T \\ -\phi_x^\dagger & \mathbf{0} & 0 \end{vmatrix}, \tag{8f}$$

and

$$f_{xx} = \begin{vmatrix} A & I & \phi_x \\ -I & B & \mathbf{0}^T \\ -\phi^\dagger & \mathbf{0} & 0 \end{vmatrix} + \begin{vmatrix} A & I & \phi \\ -I & B & \mathbf{0}^T \\ -\phi_x^\dagger & \mathbf{0} & 0 \end{vmatrix}. \tag{8g}$$

The conjugate of $g^{(s)}$ can be written as

$$g^{(s)*} = -\begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & -C_s^\dagger \\ -\phi^\dagger & \mathbf{0} & 0 \end{vmatrix}. \tag{8h}$$

Substituting for $g_x^{(s)}, g_z^{(s)}, g_{xx}^{(s)}, f_x, f_{xx}$ and f_z into equation (3a), we find

$$\begin{vmatrix} A & I & \phi & \phi_x \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & C_s & 0 & 0 \\ -\phi^\dagger & \mathbf{0} & 0 & 0 \end{vmatrix} \begin{vmatrix} A & I \\ -I & B \end{vmatrix} = \begin{vmatrix} A & I & \phi_x \\ -I & B & \mathbf{0}^T \\ -\phi^\dagger & \mathbf{0} & 0 \end{vmatrix} \begin{vmatrix} A & I & \phi \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & C_s & 0 \end{vmatrix}$$
$$- \begin{vmatrix} A & I & \phi_x \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & C_s & 0 \end{vmatrix} \begin{vmatrix} A & I & \phi \\ -I & B & \mathbf{0}^T \\ -\phi^\dagger & \mathbf{0} & 0 \end{vmatrix}. \tag{8i}$$

This is nothing but a Jacobian identity and hence $g^{(s)}$ and f satisfy the first bilinear equation (3*a*). In a similar way one can also check that the second bilinear equation (3*b*) gives rise to the following Jacobian identity for the Gram determinant forms of $g^{(s)}$ and f:

$$-\sum_{s=1}^{2}\begin{vmatrix} A & I & \phi & \mathbf{0}^T \\ -I & B & \mathbf{0}^T & -C_s^\dagger \\ -\phi^\dagger & \mathbf{0} & 0 & 0 \\ \mathbf{0} & C_s & 0 & 0 \end{vmatrix}\begin{vmatrix} A & I \\ -I & B \end{vmatrix} = -\sum_{s=1}^{2}\begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & -C_s^\dagger \\ \mathbf{0} & C_s & 0 \end{vmatrix}\begin{vmatrix} A & I & \phi \\ -I & B & \mathbf{0}^T \\ -\phi^\dagger & \mathbf{0} & 0 \end{vmatrix}$$

$$+\sum_{s=1}^{2}\begin{vmatrix} A & I & \phi \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & C_s & 0 \end{vmatrix}\begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & -C_s^\dagger \\ -\phi^\dagger & \mathbf{0} & 0 \end{vmatrix}. \qquad (8j)$$

Thus equations (8*i*) and (8*j*) clearly show that the given Gram determinants $g^{(s)}$ and f satisfy the bilinear equations (3), which complete the proof of (6) with (7).

3.2. $(N, M, N + M)$-soliton solution

We now point out that the $(N, M, N + M)$-soliton solution (for N even) given in [16] can be obtained as a special case of the above (m, m, m)-soliton solution for the specific choice of parameters $\alpha_i^{(2)} = 0, i = 1, 2, \ldots, N$ and $\alpha_l^{(1)} = 0, l = N+1, N+2, \ldots, m\ (= N+M)$ along with the parametric restrictions

$$\alpha_i^{(1)} = \frac{\prod_{j=1}^{m}\left(k_i + k_j^*\right)}{\prod_{j=1,i\neq j}^{m}(k_j - k_i)}, \qquad i = 1, 2, \ldots, N,$$

$$\alpha_l^{(2)} = \frac{\prod_{j=1}^{m}\left(k_l + k_j^*\right)}{\prod_{j=1,l\neq j}^{m}(k_j - k_l)}, \qquad l = N + 1, \quad N + 2, \ldots, m\ (= N + M).$$

In the following sections, we will consider the explicit cases of $m = 1, 2, 3$ and 4 soliton solutions and the nature of the soliton interactions therein.

4. One-soliton (1, 1, 1) and two-soliton (2, 2, 2) solutions

Specializing to the case of $m = 1$ in equation (6) so that the Gram determinants take the form

$$g^{(j)} = \begin{vmatrix} A_{11} & 1 & e^{\eta_1} \\ -1 & B_{11} & 0 \\ 0 & -\alpha_1^{(j)} & 0 \end{vmatrix}, \qquad f = \begin{vmatrix} A_{11} & 1 \\ -1 & B_{11} \end{vmatrix}, \qquad j = 1, 2, \qquad (9)$$

where $A_{11} = \frac{e^{\eta_1+\eta_1^*}}{k_1+k_1^*}$ and $B_{11} = \kappa_{11} = \frac{-(|\alpha_1^{(1)}|^2+|\alpha_1^{(2)}|^2)}{\omega_1+\omega_1^*}$, one can write the explicit one-soliton solution as

$$S^{(j)} = \frac{\alpha_1^{(j)}\, e^{\eta_1}}{1 + e^{\eta_1+\eta_1^*+R}}, \qquad j = 1, 2, \qquad (10a)$$

$$L = -2\frac{\partial^2}{\partial x^2}\left(\log\left(1 + e^{\eta_1+\eta_1^*+R}\right)\right), \qquad (10b)$$

where

$$\eta_1 = k_1 x - \left(\mathrm{i}k_1^2 + \omega_1\right)y + \omega_1 t, \qquad e^R = \frac{-\sum_{j=1}^{2}\left(\alpha_1^{(j)}\alpha_1^{(j)*}\right)}{4k_{1R}\omega_{1R}}, \qquad (10c)$$

$$k_1 = k_{1R} + \mathrm{i}k_{1I}, \qquad \omega_1 = \omega_{1R} + \mathrm{i}\omega_{1I}. \tag{10d}$$

Here $\alpha_1^{(1)}, \alpha_1^{(2)}, \omega_1$ and k_1 are all complex parameters. In equation (10) the suffixes R and I denote the real and imaginary parts, respectively. It may be noted that this bright soliton solution is nonsingular only when $k_{1R}\omega_{1R} < 0$, otherwise equation (10) becomes singular. In this work, the main focus will be on nonsingular solutions as they are of physical importance. The above one-soliton solution can also be rewritten as

$$S^{(j)} = A_j\sqrt{k_{1R}\omega_{1R}}\,\mathrm{e}^{\mathrm{i}\eta_{1I}}\,\mathrm{sech}\left(\eta_{1R} + \frac{R}{2}\right), \qquad j = 1, 2, \tag{11a}$$

$$L = -2k_{1R}^2\,\mathrm{sech}^2\left(\eta_{1R} + \frac{R}{2}\right), \tag{11b}$$

where

$$\eta_{1R} = k_{1R}x + (2k_{1R}k_{1I} - \omega_{1R})y + \omega_{1R}t \quad \text{and} \quad A_j = \frac{\alpha_1^{(j)}}{\left(\left|\alpha_1^{(1)}\right|^2 + \left|\alpha_1^{(2)}\right|^2\right)^{\frac{1}{2}}}, \qquad j = 1, 2.$$

The complex quantities $A_j\sqrt{k_{1R}\omega_{1R}}$, $j = 1, 2$, represent the amplitude of the soliton in the $S^{(j)}$ components whereas the real quantity $2k_{1R}^2$ gives the amplitude of the soliton in the component $-L$. Note that the complex quantities A_1 and A_2 satisfy the relation $|A_1|^2 + |A_2|^2 = 1$, which is a reflection of the fact that the set of equations (1) is rotationally symmetric in the $(S^{(1)}, S^{(2)})$ space.

For illustrative purposes, let us obtain the soliton solution for the special choice of parameters $\omega_1 = -\,\mathrm{i}k_1^2/2$. In this case, the above soliton solution (11) becomes

$$S^{(j)} = A_j k_{1R}\sqrt{k_{1I}}\,\mathrm{e}^{\mathrm{i}\eta_{1I}}\,\mathrm{sech}\left(\eta_{1R} + \frac{R}{2}\right), \qquad j = 1, 2, \tag{12a}$$

$$L = -2k_{1R}^2\,\mathrm{sech}^2\left(\eta_{1R} + \frac{R}{2}\right), \tag{12b}$$

where

$$\eta_1 = k_1x - \frac{\mathrm{i}k_1^2}{2}(t + y), \qquad \mathrm{e}^R = \frac{\sum_{j=1}^2\left(\alpha_1^{(j)}\alpha_1^{(j)*}\right)}{-4k_{1R}^2k_{1I}}, \qquad k_1 = k_{1R} + \mathrm{i}k_{1I}. \tag{12c}$$

The above soliton solution is nonsingular only when $k_{1I} \leqslant 0$, otherwise the parameter R in equation (12*c*) becomes complex and the solution (12) becomes singular. Interestingly, we observe that by just reducing the width of the soliton in the L component (which is proportional to k_{1I}) without affecting its amplitude, the soliton in the $S^{(1)}$ and $S^{(2)}$ components can be amplified with a proportionate pulse compression, a desirable property for a pulse in nonlinear optics.

4.1. *Two-soliton (2, 2, 2) solution*

To obtain the two-soliton solution, we take $m = 2$ in equation (7) and deduce the Gram determinant forms as

$$g^{(j)} = \begin{vmatrix} A_{11} & A_{12} & 1 & 0 & \mathrm{e}^{\eta_1} \\ A_{21} & A_{22} & 0 & 1 & \mathrm{e}^{\eta_2} \\ -1 & 0 & B_{11} & B_{12} & 0 \\ 0 & -1 & B_{21} & B_{22} & 0 \\ 0 & 0 & -\alpha_1^{(j)} & -\alpha_2^{(j)} & 0 \end{vmatrix}, \qquad f = \begin{vmatrix} A_{11} & A_{12} & 1 & 0 \\ A_{21} & A_{22} & 0 & 1 \\ -1 & 0 & B_{11} & B_{12} \\ 0 & -1 & B_{21} & B_{22} \end{vmatrix}, \tag{13}$$

where $A_{ij} = \frac{e^{\eta_i+\eta_j^*}}{k_i+k_j^*}$ and $B_{ij} = \kappa_{ji} = -\frac{(\alpha_j^{(1)}\alpha_i^{(1)*}+\alpha_j^{(2)}\alpha_i^{(2)*})}{(\omega_j+\omega_i^*)}$, $i, j = 1, 2$. We can then write the explicit form of the (2, 2, 2) soliton solution as

$$S^{(j)} = \frac{1}{f}\left(\alpha_1^{(j)}\,e^{\eta_1} + \alpha_2^{(j)}\,e^{\eta_2} + e^{\eta_1+\eta_1^*+\eta_2+\delta_{1j}} + e^{\eta_2+\eta_2^*+\eta_1+\delta_{2j}}\right), \qquad j = 1, 2, \tag{14a}$$

$$L = -2\frac{\partial^2}{\partial x^2}\log(f), \tag{14b}$$

where

$$f = 1 + e^{\eta_1+\eta_1^*+R_1} + e^{\eta_1+\eta_2^*+\delta_0} + e^{\eta_2+\eta_1^*+\delta_0^*} + e^{\eta_2+\eta_2^*+R_2} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_3}. \tag{14c}$$

The various quantities found in equation (14) are defined as below:

$$\eta_i = k_i x - \left(\mathrm{i}k_i^2 + \omega_i\right)y + \omega_i t, \qquad i = 1, 2, \qquad e^{R_1} = \frac{\kappa_{11}}{\left(k_1 + k_1^*\right)}, \tag{15a}$$

$$e^{R_2} = \frac{\kappa_{22}}{\left(k_2 + k_2^*\right)}, \qquad e^{\delta_0} = \frac{\kappa_{12}}{\left(k_1 + k_2^*\right)}, \qquad e^{\delta_0^*} = \frac{\kappa_{21}}{\left(k_2 + k_1^*\right)}, \tag{15b}$$

$$e^{\delta_{1j}} = \frac{(k_1 - k_2)}{\left(k_1 + k_1^*\right)\left(k_2 + k_1^*\right)}\left(\alpha_1^{(j)}\kappa_{21} - \alpha_2^{(j)}\kappa_{11}\right), \tag{15c}$$

$$e^{\delta_{2j}} = \frac{(k_2 - k_1)}{\left(k_2 + k_2^*\right)\left(k_1 + k_2^*\right)}\left(\alpha_2^{(j)}\kappa_{12} - \alpha_1^{(j)}\kappa_{22}\right), \qquad j = 1, 2, \tag{15d}$$

$$e^{R_3} = \frac{|k_1 - k_2|^2}{\left(k_1 + k_1^*\right)\left(k_2 + k_2^*\right)\left|k_1 + k_2^*\right|^2}(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}), \tag{15e}$$

$$\kappa_{il} = -\frac{\left(\alpha_i^{(1)}\alpha_l^{(1)*} + \alpha_i^{(2)}\alpha_l^{(2)*}\right)}{\left(\omega_i + \omega_l^*\right)}, \qquad i, l = 1, 2.$$

The two-soliton solution is characterized by eight arbitrary complex parameters $\alpha_1^{(1)}, \alpha_1^{(2)}, \alpha_2^{(1)}, \alpha_2^{(2)}, k_1, k_2, \omega_1$ and ω_2. The above solution features both singular and nonsingular solutions. The nonsingular solution can be obtained by requiring the denominator f in (14) to be real and nonzero. The expression (14*c*) for f can be rewritten as

$$f = 2\,e^{\eta_{1R}+\eta_{2R}}(e^{(R_1+R_2)/2}\cosh(\eta_{1R} - \eta_{2R} + (R_1 + R_2)/2) + e^{\delta_{0R}}\cos(\eta_{1I} - \eta_{2I} + \delta_{0I}) + e^{R_3/2}\cosh(\eta_{1R} + \eta_{2R} + R_3/2)). \tag{15f}$$

To get regular solutions, e^{R_1} and e^{R_2} should be positive which can be obtained only for $k_{1R}\omega_{1R} < 0$ and $k_{2R}\omega_{2R} < 0$, respectively. Otherwise, that is for negative values, the solution is not regular as in this case R_1 and R_2 appearing in the argument of cosh in first term become complex. So the condition $k_{jR}\omega_{jR} < 0$, $j = 1, 2$ is a necessary condition to obtain a regular solution. In a similar way, in the third term, the quantity $R_3/2$ becomes real and positive for the condition $\kappa_{11}\kappa_{22} - |\kappa_{12}|^2 > 0$, as may be seen from equation (15e). Still the middle term $\cos(\eta_{1I} - \eta_{2I} + \delta_{0I})$ can lead to a singularity as it oscillates between −1 and 1. This can be eliminated by choosing the coefficients of the remaining two terms as $e^{(R_1+R_2)/2} + e^{R_3/2} > e^{\delta_{0R}}$, in order to ensure that f will not be zero at any point in space and time. The last condition is a sufficient one. As an illustration, the interaction of two solitons in system (1) is shown in figure 1. The parameters are chosen as $k_1 = 1 - 2\,\mathrm{i}$, $k_2 = 1.5 - 1.05\,\mathrm{i}$, $\omega_1 = -1 - \mathrm{i}$, $\omega_2 = -1.3 - 0.5\,\mathrm{i}$, $\alpha_1^{(1)} = 2$, $\alpha_2^{(1)} = \alpha_1^{(2)} = 1$, $\alpha_2^{(2)} = 0.01$. One observes that the solitons in the $S^{(1)}$ and $S^{(2)}$ components undergo shape changing (energy redistribution) collisions while there is only an elastic collision in the L component. More details are given in section 6.

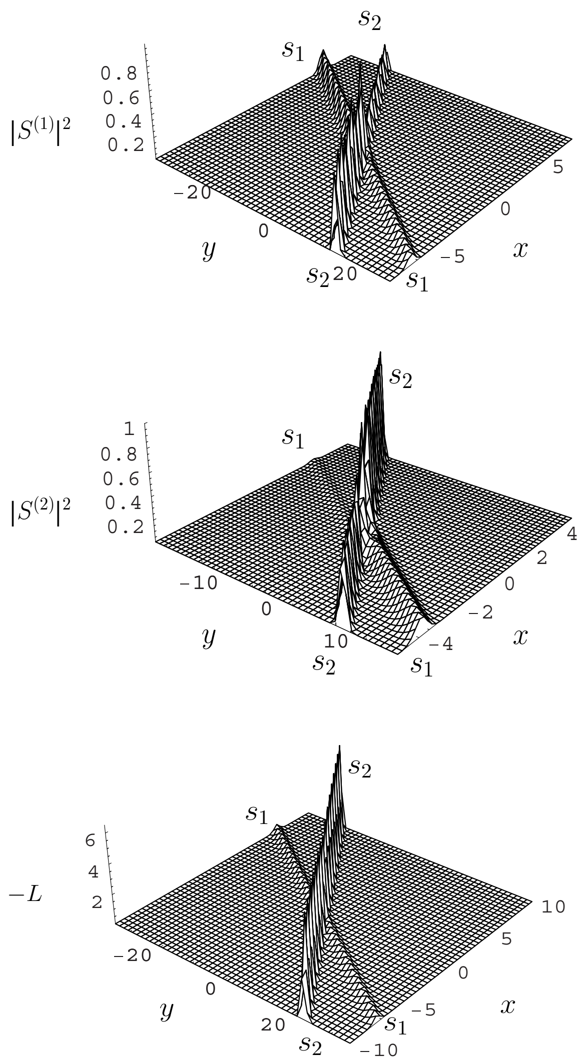

Figure 1. Shape changing collision of solitons in the three-wave system with the parametric choice $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} \neq \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$. Here the two-soliton solution (14) is plotted for a fixed value of t and the associated parameters are given in the text (below equation (15)). Note that intensity redistribution occurs only in the $S^{(1)}$ and $S^{(2)}$ components, while elastic collision only occurs in the L component.

4.2. (1, 1, 2)-soliton solution of Ohta et al

Now we show that the (1, 1, 2)-soliton solution obtained by Ohta *et al* [16] is a special case of the above two-soliton (2, 2, 2) solution (14). Specifically, for the special choice of the parameters $\alpha_2^{(1)} = \alpha_1^{(2)} = 0$, the above two-soliton solution becomes

$$S^{(1)} = \frac{1}{f}\left(\alpha_1^{(1)} e^{\eta_1} + e^{\eta_2+\eta_2^*+\eta_1+\delta_{21}}\right), \tag{16a}$$

$$S^{(2)} = \frac{1}{f}\left(\alpha_2^{(2)}\,\mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\delta_{12}}\right), \tag{16b}$$

$$L = -2\frac{\partial^2}{\partial x^2}(\log(f)), \tag{16c}$$

where

$$f = 1 + \mathrm{e}^{\eta_1+\eta_1^*+R_1} + \mathrm{e}^{\eta_2+\eta_2^*+R_2} + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_3}. \tag{16d}$$

The various other parameters defined in equations (14) now take the forms

$$\mathrm{e}^{R_1} = \frac{\kappa_{11}}{(k_1+k_1^*)}, \qquad \mathrm{e}^{R_2} = \frac{\kappa_{22}}{(k_2+k_2^*)}, \qquad \mathrm{e}^{\delta_0} = \mathrm{e}^{\delta_{11}} = \mathrm{e}^{\delta_{22}} = 0, \tag{16e}$$

$$\mathrm{e}^{\delta_{12}} = \frac{-\alpha_2^{(2)}\kappa_{11}(k_1-k_2)}{(k_1+k_1^*)(k_2+k_1^*)}, \qquad \mathrm{e}^{\delta_{21}} = \frac{-\alpha_1^{(1)}\kappa_{22}(k_2-k_1)}{(k_2+k_2^*)(k_1+k_2^*)}, \tag{16f}$$

$$\mathrm{e}^{R_3} = \frac{|k_1-k_2|^2\kappa_{11}\kappa_{22}}{(k_1+k_1^*)(k_2+k_2^*)\left|k_1+k_2^*\right|^2}, \tag{16g}$$

$$\kappa_{11} = -\frac{|\alpha_1^{(1)}|^2}{(\omega_1+\omega_1^*)}, \qquad \kappa_{22} = -\frac{|\alpha_2^{(2)}|^2}{(\omega_2+\omega_2^*)}. \tag{16h}$$

Solutions (16*a*)–(16*h*) are nothing but the (1, 1, 2)-soliton solution obtained by Ohta *et al* in [16] when the parameters in (16*a*)–(16*h*) are further restricted to the special choice

$$\alpha_1^{(1)} = \frac{(k_1+k_1^*)(k_1+k_2^*)}{(k_2-k_1)} \qquad \text{and} \qquad \alpha_2^{(2)} = \frac{(k_2+k_2^*)(k_2+k_1^*)}{(k_2-k_1)}. \tag{16i}$$

5. Asymptotic analysis of the two-soliton solution (14) of the three-wave system

We now consider the collision properties associated with the general two-soliton solution (14) of the three-wave system. For this purpose we carry out the analysis, for $k_{jR} > 0, \omega_{jR} < 0, j = 1, 2$. Also we choose $\frac{k_{2R}}{k_{1R}} > \left|\frac{\omega_{2R}}{\omega_{1R}}\right|$ and $\frac{k_{2R}k_{2I}}{k_{1R}k_{1I}} > \left|\frac{\omega_{2R}}{\omega_{1R}}\right|$ for convenience. Similar analysis can be performed for other choices of k_{jR} and ω_{jR} also by keeping $k_{jR} > 0, \omega_{jR} < 0$, which is the necessary condition for nonsingular solutions. We now define the soliton wave variables as $\eta_{1R} = k_{1R}x + (2k_{1R}k_{1I} - \omega_{1R})y + \omega_{1R}t$ and $\eta_{2R} = k_{2R}x + (2k_{2R}k_{2I} - \omega_{2R})y + \omega_{2R}t$. In the limit $x, y \to \pm\infty$ and a fixed t the two-soliton solution (14) takes the following asymptotic forms.

(a) *Before collision* (*limit* $x, y \to -\infty$):

(i) soliton 1 ($\eta_{1R} \simeq 0, \eta_{2R} \to -\infty$):

$$\begin{pmatrix} S^{(1)} \\ S^{(2)} \end{pmatrix} \simeq \begin{pmatrix} A_1^{1-} \\ A_2^{1-} \end{pmatrix} \sqrt{k_{1R}\omega_{1R}}\,\mathrm{sech}\left(\eta_{1R} + \frac{R_1}{2}\right)\mathrm{e}^{\mathrm{i}\eta_{1I}}, \tag{17a}$$

$$L \simeq -2k_{1R}^2\,\mathrm{sech}^2\left(\eta_{1R} + \frac{R_1}{2}\right), \tag{17b}$$

where

$$\begin{pmatrix} A_1^{1-} \\ A_2^{1-} \end{pmatrix} \simeq \begin{pmatrix} \alpha_1^{(1)} \\ \alpha_1^{(2)} \end{pmatrix} \frac{\mathrm{e}^{-R_1/2}}{\left((k_1+k_1^*)(\omega_1+\omega_1^*)\right)^{1/2}}. \tag{17c}$$

(ii) soliton 2 ($\eta_{2R} \simeq 0, \eta_{1R} \to \infty$):

$$\begin{pmatrix} S^{(1)} \\ S^{(2)} \end{pmatrix} \simeq \begin{pmatrix} A_1^{2-} \\ A_2^{2-} \end{pmatrix} \sqrt{k_{2R}\omega_{2R}}\,\mathrm{sech}\left(\eta_{2R} + \frac{(R_3 - R_1)}{2}\right) \mathrm{e}^{\mathrm{i}\eta_{2I}}, \tag{18a}$$

$$L \simeq -2k_{2R}^2\,\mathrm{sech}^2\left(\eta_{2R} + \frac{(R_3 - R_1)}{2}\right), \tag{18b}$$

where

$$\begin{pmatrix} A_1^{2-} \\ A_2^{2-} \end{pmatrix} \simeq \begin{pmatrix} \mathrm{e}^{\delta_{11}} \\ \mathrm{e}^{\delta_{12}} \end{pmatrix} \frac{\mathrm{e}^{-(R_1+R_3)/2}}{\left(\left(k_2 + k_2^*\right)\left(\omega_2 + \omega_2^*\right)\right)^{1/2}}. \tag{18c}$$

The various quantities in the above expressions are defined in equation (15).

(b) *After collision* (*limit* $x, y \to \infty$):

(i) soliton 1 ($\eta_{1R} \simeq 0, \eta_{2R} \to \infty$):

$$\begin{pmatrix} S^{(1)} \\ S^{(2)} \end{pmatrix} \simeq \begin{pmatrix} A_1^{1+} \\ A_2^{1+} \end{pmatrix} \sqrt{k_{1R}\omega_{1R}}\,\mathrm{sech}\left(\eta_{1R} + \frac{(R_3 - R_2)}{2}\right) \mathrm{e}^{\mathrm{i}\eta_{1I}}, \tag{19a}$$

$$L \simeq -2k_{1R}^2\,\mathrm{sech}^2\left(\eta_{1R} + \frac{(R_3 - R_2)}{2}\right), \tag{19b}$$

where

$$\begin{pmatrix} A_1^{1+} \\ A_2^{1+} \end{pmatrix} \simeq \begin{pmatrix} \mathrm{e}^{\delta_{21}} \\ \mathrm{e}^{\delta_{22}} \end{pmatrix} \frac{\mathrm{e}^{-(R_2+R_3)/2}}{\left(\left(k_1 + k_1^*\right)\left(\omega_1 + \omega_1^*\right)\right)^{1/2}}. \tag{19c}$$

(ii) soliton 2 ($\eta_{2R} \simeq 0, \eta_{1R} \to -\infty$):

$$\begin{pmatrix} S^{(1)} \\ S^{(2)} \end{pmatrix} \simeq \begin{pmatrix} A_1^{2+} \\ A_2^{2+} \end{pmatrix} \sqrt{k_{2R}\omega_{2R}}\,\mathrm{sech}\left(\eta_{2R} + \frac{R_2}{2}\right) \mathrm{e}^{\mathrm{i}\eta_{2I}}, \tag{20a}$$

$$L \simeq -2k_{2R}^2\,\mathrm{sech}^2\left(\eta_{2R} + \frac{R_2}{2}\right), \tag{20b}$$

where

$$\begin{pmatrix} A_1^{2+} \\ A_2^{2+} \end{pmatrix} \simeq \begin{pmatrix} \alpha_2^{(1)} \\ \alpha_2^{(2)} \end{pmatrix} \frac{\mathrm{e}^{-R_2/2}}{\left(\left(k_2 + k_2^*\right)(\omega_2 + \omega_2^*)\right)^{1/2}}. \tag{20c}$$

Note that in all the above expressions $\left|A_1^{j\pm}\right|^2 + \left|A_2^{j\pm}\right|^2 = 1, j = 1, 2$. Our above analysis reveals the fact that due to collision the amplitudes of the colliding solitons, say s_1 and s_2 in the $S^{(1)}$ and $S^{(2)}$ components, change from $\left(A_1^{1-}, A_2^{1-}\right)\sqrt{k_{1R}\omega_{1R}}$ and $\left(A_1^{2-}, A_2^{2-}\right)\sqrt{k_{2R}\omega_{2R}}$ to $\left(A_1^{1+}, A_2^{1+}\right)\sqrt{k_{1R}\omega_{1R}}$ and $\left(A_1^{2+}, A_2^{2+}\right)\sqrt{k_{2R}\omega_{2R}}$, respectively. Here the superscripts in $A_i^{j\pm}$ with $i, j = 1, 2$ denote the solitons s_1 and s_2, while the subscripts represent the components $S^{(1)}$ and $S^{(2)}$ and the '$\pm$' signs stand for '$x, y \to \pm\infty$'. In addition to this change in the amplitudes, the solitons also undergo amplitude-dependent phase shifts due to the collision, and they can be determined straightforwardly from the above asymptotic expressions. From equations (17) and (19) and equations (18) and (20), one can easily check that the phase shift suffered by the soliton s_1 (say Φ_1) = − phase shift of soliton s_2 (say $-\Phi_2 \equiv \Phi_1$) = Φ and is given by

$$\Phi = \frac{(R_3 - R_1 - R_2)}{2}, \tag{21}$$

where R_1, R_2 and R_3 are as defined in equation (15) and depend on the amplitudes.

6. Soliton interaction

Now it is of further interest to analyse the interaction properties of the solitons depicted in figure 1 for the specific set of values of the parameters given in section 4.1. Figure 1 shows a typical spatial collision of two solitons for $t = -4$ corresponding to the exact expression (14). The interesting collision scenario depicted in figure 1 clearly indicates that there is a redistribution of intensity among the two $S^{(j)}$ components resulting in an enhancement (suppression) of intensity of solitons s_2 (s_1) in the $S^{(1)}$ component and a suppression (enhancement) of soliton s_2 (s_1) in the $S^{(2)}$ component. The solitons also undergo amplitude-dependent phase shifts along with this energy redistribution. However, the solitons appearing in the long-wave component (L) exhibit the standard elastic collision as shown in the third figure of figure 1 though the phase shift here is also amplitude dependent. Interestingly, if the parameters are chosen such that the condition $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} = \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$ is satisfied, there occurs only an elastic collision in all the three components, $S^{(j)}$ and L. The underlying collision dynamics can be well understood by using the asymptotic analysis of the two-soliton solution (14) discussed in section 5 and is further described below.

6.1. Collision behaviour of solitons in the short-wave components

The asymptotic analysis presented in the previous section also results in the following expressions relating the intensities of solitons s_1 and s_2 in the $S^{(1)}$ and $S^{(2)}$ components before and after the interaction,

$$\left|A_i^{j+}\right|^2 = \left|T_j^i\right|^2 \left|A_i^{j-}\right|^2, \qquad i, j = 1, 2, \tag{22a}$$

where the superscripts $j\pm$ represent the solitons designated as s_1 and s_2 at '$x, y \to \pm\infty$'. The expression for the transition intensities for the solitons in the short-wave components can be written using the results in equations (17c), (18c), (19c) and (20c) as

$$\left|T_j^1\right|^2 = \frac{\left|1 - \lambda_2\left(\alpha_2^{(j)}/\alpha_1^{(j)}\right)\right|^2}{|1 - \lambda_1\lambda_2|}, \tag{22b}$$

$$\left|T_j^2\right|^2 = \frac{|1 - \lambda_1\lambda_2|}{\left|1 - \lambda_1\left(\alpha_1^{(j)}/\alpha_2^{(j)}\right)\right|^2}, \qquad j = 1, 2, \tag{22c}$$

$$\lambda_1 = \frac{\kappa_{21}}{\kappa_{11}}, \qquad \lambda_2 = \frac{\kappa_{12}}{\kappa_{22}}. \tag{22d}$$

In general, $\left|T_j^i\right| \neq 1$ and so an intensity (energy) redistribution of the solitons in the $S^{(1)}$ and $S^{(2)}$ components occurs as shown in figure 1. One can note that the standard elastic collision takes place for the specific parametric choice $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} = \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$, as $\left|T_j^i\right|^2 = 1$ and hence $\left|A_i^{j-}\right|^2 = \left|A_i^{j+}\right|^2$, $i, j = 1, 2$ for this choice. However, the two colliding solitons s_1 and s_2 suffer amplitude-dependent phase shifts Φ_1 and Φ_2, respectively, as given in equation (21).

6.2. Collision scenario in the long-wave component

In the L component, there occurs only elastic collision for any parametric choice. This is evident from the asymptotic analysis, see equations (17b), (18b), (19b) and (20b). One finds that the amplitudes of the solitons s_1 and s_2 before and after interaction are the same, i.e. $-2k_{1R}^2$ and $-2k_{2R}^2$, respectively, while there occurs an amplitude-dependent phase shift as given by equation (21).

6.3. Shape changing collisions and linear fractional transformations

It is instructive to note that the intensity redistribution in the short-wave components characterized by the transition matrices (equation (22)) can also be viewed as a linear fractional transformation (LFT). To realize this, we re-express the amplitude changes in the short-wave components of soliton s_1 after interaction as

$$A_1^{1+} = \Gamma C_{11} A_1^{1-} + \Gamma C_{12} A_2^{1-}, \tag{23a}$$

$$A_2^{1+} = \Gamma C_{21} A_1^{1-} + \Gamma C_{22} A_2^{1-}. \tag{23b}$$

Here,

$$\Gamma = \left(\frac{a}{a^*}\right) c\left[\left(\alpha_1^{(1)}\alpha_2^{(1)*} + \alpha_1^{(2)}\alpha_2^{(2)*}\right)\left(\alpha_2^{(1)}\alpha_2^{(1)*} + \alpha_2^{(2)}\alpha_2^{(2)*}\right)\right]^{-1}, \tag{23c}$$

$$C_{11} = -\left[\left(\alpha_2^{(1)}\alpha_2^{(1)*}\right)(\omega_1 - \omega_2) + \left(\alpha_2^{(2)}\alpha_2^{(2)*}\right)\left(\omega_1 + \omega_2^*\right)\right], \tag{23d}$$

$$C_{12} = \left(\alpha_2^{(1)}\alpha_2^{(2)*}\right)\left(\omega_2 + \omega_2^*\right), \tag{23e}$$

$$C_{21} = \left(\alpha_2^{(2)}\alpha_2^{(1)*}\right)\left(\omega_2 + \omega_2^*\right), \tag{23f}$$

$$C_{22} = -\left[\left(\alpha_2^{(1)}\alpha_2^{(1)*}\right)\left(\omega_1 + \omega_2^*\right) + \left(\alpha_2^{(2)}\alpha_2^{(2)*}\right)(\omega_1 - \omega_2)\right], \tag{23g}$$

where

$$c = \left(\frac{1}{|\kappa_{12}|^2} - \frac{1}{\kappa_{11}\kappa_{22}}\right)^{-1/2}, \tag{23h}$$

$$a = \left[-(k_1 - k_2)\left(k_2 + k_1^*\right)\left(\omega_2 + \omega_1^*\right)\left(\alpha_1^{(1)}\alpha_2^{(1)*} + \alpha_1^{(2)}\alpha_2^{(2)*}\right)\right]^{1/2}. \tag{23i}$$

Note that the coefficients C_{ij} s, $i, j = 1, 2$, are independent of $\alpha_1^{(j)}$ s and so of A_1^{1-} and A_2^{1-}, that is the α parameters of soliton s_1. From equations (23a) and (23b),

$$\rho_{1,2}^{1+} = \frac{A_1^{1+}}{A_2^{1+}} = \frac{C_{11}\rho_{1,2}^{1-} + C_{12}}{C_{21}\rho_{1,2}^{1-} + C_{22}}, \tag{23j}$$

where $\rho_{1,2}^{1-} = \frac{A_1^{1-}}{A_2^{1-}}$, in which the superscripts represent the underlying soliton and the subscripts represent the corresponding short-wave components. Thus the state of s_1 before and after interaction is characterized by the complex quantities $\rho_{1,2}^{1-}$ and $\rho_{1,2}^{1+}$, respectively. The direct consequence of the above LFT representation is the identification of a binary logic using soliton collisions as in the case of CNLS equations [6, 14, 15] and hence the LFT can profitably be used for constructing logic gates associated with the binary logic. A similar analysis can also be made for the soliton s_2.

7. Three-soliton (3, 3, 3) solution

From the general form (7), and restricting $m = 3$, one can write the explicit three-soliton (3, 3, 3) solution as

$$S^{(j)} = \frac{\alpha_1^{(j)}\,\mathrm{e}^{\eta_1} + \alpha_2^{(j)}\,\mathrm{e}^{\eta_2} + \alpha_3^{(j)}\,\mathrm{e}^{\eta_3} + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\delta_{1j}} + \mathrm{e}^{\eta_1+\eta_1^*+\eta_3+\delta_{2j}} + \mathrm{e}^{\eta_2+\eta_2^*+\eta_1+\delta_{3j}}}{f}$$
$$+ \frac{\mathrm{e}^{\eta_2+\eta_2^*+\eta_3+\delta_{4j}} + \mathrm{e}^{\eta_3+\eta_3^*+\eta_1+\delta_{5j}} + \mathrm{e}^{\eta_3+\eta_3^*+\eta_2+\delta_{6j}} + \mathrm{e}^{\eta_1^*+\eta_2+\eta_3+\delta_{7j}} + \mathrm{e}^{\eta_1+\eta_2^*+\eta_3+\delta_{8j}}}{f}$$

$$+\frac{e^{\eta_1+\eta_2+\eta_3^*+\delta_{9j}}+e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\eta_3+\tau_{1j}}+e^{\eta_1+\eta_1^*+\eta_3+\eta_3^*+\eta_2+\tau_{2j}}}{f}$$

$$+\frac{e^{\eta_2+\eta_2^*+\eta_3+\eta_3^*+\eta_1+\tau_{3j}}}{f}, \qquad j=1,2, \tag{24a}$$

where

$$\begin{aligned}f = 1 &+ e^{\eta_1+\eta_1^*+R_1}+e^{\eta_2+\eta_2^*+R_2}+e^{\eta_3+\eta_3^*+R_3}+e^{\eta_1+\eta_2^*+\delta_{10}}+e^{\eta_1^*+\eta_2+\delta_{10}^*}\\ &+e^{\eta_1+\eta_3^*+\delta_{20}}+e^{\eta_1^*+\eta_3+\delta_{20}^*}+e^{\eta_2+\eta_3^*+\delta_{30}}+e^{\eta_2^*+\eta_3+\delta_{30}^*}+e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_4}\\ &+e^{\eta_1+\eta_1^*+\eta_3+\eta_3^*+R_5}+e^{\eta_2+\eta_2^*+\eta_3+\eta_3^*+R_6}+e^{\eta_1+\eta_1^*+\eta_2+\eta_3^*+\tau_{10}}+e^{\eta_1+\eta_1^*+\eta_3+\eta_2^*+\tau_{10}^*}\\ &+e^{\eta_2+\eta_2^*+\eta_1+\eta_3^*+\tau_{20}}+e^{\eta_2+\eta_2^*+\eta_1^*+\eta_3+\tau_{20}^*}+e^{\eta_3+\eta_3^*+\eta_1+\eta_2^*+\tau_{30}}+e^{\eta_3+\eta_3^*+\eta_1^*+\eta_2+\tau_{30}^*}\\ &+e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\eta_3+\eta_3^*+R_7}.\end{aligned} \tag{24b}$$

Here,

$$\eta_i = k_i x - \left(ik_i^2+\omega_i\right)y+\omega_i t, \qquad i=1,2,3,$$

$$e^{\delta_{1j}} = \frac{(k_1-k_2)\left(\alpha_1^{(j)}\kappa_{21}-\alpha_2^{(j)}\kappa_{11}\right)}{\left(k_1+k_1^*\right)\left(k_1^*+k_2\right)}, \qquad e^{\delta_{2j}} = \frac{(k_1-k_3)\left(\alpha_1^{(j)}\kappa_{31}-\alpha_3^{(j)}\kappa_{11}\right)}{\left(k_1+k_1^*\right)\left(k_1^*+k_3\right)},$$

$$e^{\delta_{3j}} = \frac{(k_1-k_2)\left(\alpha_1^{(j)}\kappa_{22}-\alpha_2^{(j)}\kappa_{12}\right)}{\left(k_1+k_2^*\right)\left(k_2+k_2^*\right)}, \qquad e^{\delta_{4j}} = \frac{(k_2-k_3)\left(\alpha_2^{(j)}\kappa_{32}-\alpha_3^{(j)}\kappa_{22}\right)}{\left(k_2+k_2^*\right)\left(k_2^*+k_3\right)},$$

$$e^{\delta_{5j}} = \frac{(k_1-k_3)\left(\alpha_1^{(j)}\kappa_{33}-\alpha_3^{(j)}\kappa_{13}\right)}{\left(k_3+k_3^*\right)\left(k_3^*+k_1\right)}, \qquad e^{\delta_{6j}} = \frac{(k_2-k_3)\left(\alpha_2^{(j)}\kappa_{33}-\alpha_3^{(j)}\kappa_{23}\right)}{\left(k_3^*+k_2\right)\left(k_3^*+k_3\right)},$$

$$e^{\delta_{7j}} = \frac{(k_2-k_3)\left(\alpha_2^{(j)}\kappa_{31}-\alpha_3^{(j)}\kappa_{21}\right)}{\left(k_1^*+k_2\right)\left(k_1^*+k_3\right)}, \qquad e^{\delta_{8j}} = \frac{(k_1-k_3)\left(\alpha_1^{(j)}\kappa_{32}-\alpha_3^{(j)}\kappa_{12}\right)}{\left(k_1+k_2^*\right)\left(k_2^*+k_3\right)},$$

$$e^{\delta_{9j}} = \frac{(k_1-k_2)\left(\alpha_1^{(j)}\kappa_{23}-\alpha_2^{(j)}\kappa_{13}\right)}{\left(k_1+k_3^*\right)\left(k_2+k_3^*\right)},$$

$$\begin{aligned}e^{\tau_{1j}} &= \frac{(k_2-k_1)(k_3-k_1)(k_3-k_2)\left(k_2^*-k_1^*\right)}{\left(k_1^*+k_1\right)\left(k_1^*+k_2\right)\left(k_1^*+k_3\right)\left(k_2^*+k_1\right)\left(k_2^*+k_2\right)\left(k_2^*+k_3\right)}\\ &\quad\times\left[\alpha_1^{(j)}(\kappa_{21}\kappa_{32}-\kappa_{22}\kappa_{31})+\alpha_2^{(j)}(\kappa_{12}\kappa_{31}-\kappa_{32}\kappa_{11})+\alpha_3^{(j)}(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21})\right],\\ e^{\tau_{2j}} &= \frac{(k_2-k_1)(k_3-k_1)(k_3-k_2)\left(k_3^*-k_1^*\right)}{\left(k_1^*+k_1\right)\left(k_1^*+k_2\right)\left(k_1^*+k_3\right)\left(k_3^*+k_1\right)\left(k_3^*+k_2\right)\left(k_3^*+k_3\right)}\\ &\quad\times\left[\alpha_1^{(j)}(\kappa_{33}\kappa_{21}-\kappa_{31}\kappa_{23})+\alpha_2^{(j)}(\kappa_{31}\kappa_{13}-\kappa_{11}\kappa_{33})+\alpha_3^{(j)}(\kappa_{23}\kappa_{11}-\kappa_{13}\kappa_{21})\right],\\ e^{\tau_{3j}} &= \frac{(k_2-k_1)(k_3-k_1)(k_3-k_2)\left(k_3^*-k_2^*\right)}{\left(k_2^*+k_1\right)\left(k_2^*+k_2\right)\left(k_2^*+k_3\right)\left(k_3^*+k_1\right)\left(k_3^*+k_2\right)\left(k_3^*+k_3\right)}\\ &\quad\times\left[\alpha_1^{(j)}(\kappa_{22}\kappa_{33}-\kappa_{23}\kappa_{32})+\alpha_2^{(j)}(\kappa_{13}\kappa_{32}-\kappa_{33}\kappa_{12})+\alpha_3^{(j)}(\kappa_{12}\kappa_{23}-\kappa_{22}\kappa_{13})\right],\end{aligned} \tag{24d}$$

$$e^{R_m} = \frac{\kappa_{mm}}{k_m+k_m^*}, \quad m=1,2,3, \quad e^{\delta_{10}} = \frac{\kappa_{12}}{k_1+k_2^*}, \quad e^{\delta_{20}} = \frac{\kappa_{13}}{k_1+k_3^*}, \quad e^{\delta_{30}} = \frac{\kappa_{23}}{k_2+k_3^*},$$

$$e^{R_4} = \frac{(k_2 - k_1)(k_2^* - k_1^*)}{(k_1^* + k_1)(k_1^* + k_2)(k_1 + k_2^*)(k_2^* + k_2)} [\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}],$$

$$e^{R_5} = \frac{(k_3 - k_1)(k_3^* - k_1^*)}{(k_1^* + k_1)(k_1^* + k_3)(k_3^* + k_1)(k_3^* + k_3)} [\kappa_{33}\kappa_{11} - \kappa_{13}\kappa_{31}],$$

$$e^{R_6} = \frac{(k_3 - k_2)(k_3^* - k_2^*)}{(k_2^* + k_2)(k_2^* + k_3)(k_3^* + k_2)(k_3 + k_3^*)} [\kappa_{22}\kappa_{33} - \kappa_{23}\kappa_{32}],$$

$$e^{\tau_{10}} = \frac{(k_2 - k_1)(k_3^* - k_1^*)}{(k_1^* + k_1)(k_1^* + k_2)(k_3^* + k_1)(k_3^* + k_2)} [\kappa_{11}\kappa_{23} - \kappa_{21}\kappa_{13}],$$

$$e^{\tau_{20}} = \frac{(k_1 - k_2)(k_3^* - k_2^*)}{(k_2^* + k_1)(k_2^* + k_2)(k_3^* + k_1)(k_3^* + k_2)} [\kappa_{22}\kappa_{13} - \kappa_{12}\kappa_{23}],$$

$$e^{\tau_{30}} = \frac{(k_3 - k_1)(k_3^* - k_2^*)}{(k_2^* + k_1)(k_2^* + k_3)(k_3^* + k_1)(k_3^* + k_3)} [\kappa_{33}\kappa_{12} - \kappa_{13}\kappa_{32}],$$

$$e^{R_7} = \frac{|k_1 - k_2|^2|k_2 - k_3|^2|k_3 - k_1|^2}{(k_1 + k_1^*)(k_2 + k_2^*)(k_3 + k_3^*)|k_1 + k_2^*|^2|k_2 + k_3^*|^2|k_3 + k_1^*|^2} \times [(\kappa_{11}\kappa_{22}\kappa_{33} - \kappa_{11}\kappa_{23}\kappa_{32}) + (\kappa_{12}\kappa_{23}\kappa_{31} - \kappa_{12}\kappa_{21}\kappa_{33}) + (\kappa_{21}\kappa_{13}\kappa_{32} - \kappa_{22}\kappa_{13}\kappa_{31})], \tag{24e}$$

and

$$\kappa_{il} = -\frac{\sum_{n=1}^{2} \alpha_i^{(n)}\alpha_l^{(n)*}}{(\omega_i + \omega_l^*)}, \qquad i, l = 1, 2, 3. \tag{24f}$$

The explicit form of L can be obtained by substituting the expression for f into $L = -2\frac{\partial^2}{\partial x^2}(\log f)$. Here $\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}, \alpha_1^{(2)}, \alpha_2^{(2)}, \alpha_3^{(2)}, k_1, k_2, k_3, \omega_1, \omega_2$ and ω_3 are the 12 complex parameters which characterize the above three-soliton solution. Following the arguments of [8] (see equations (28) and (29)) and the discussion in section 4 one can show that the necessary conditions for the nonsingular solution are

$$e^{R_i} > 0, \qquad i = 1, 2, \ldots 7, \tag{24g}$$

which is automatically taken care of by the choice $k_{jR}\omega_{jR} < 0$, $j = 1, 2, 3$. One can also easily show that the inequality

$$e^{(R_1+R_6)/2}, \quad e^{(R_2+R_5)/2}, \quad e^{(R_3+R_4)/2}, \quad e^{(R_7)/2} > 4\max(e^{(\delta_{10R}+\tau_{30R})}, e^{(\delta_{20R}+\tau_{20R})}, e^{(\delta_{30R}+\tau_{10R})}) \tag{24h}$$

is the sufficient condition in order to ensure that the solution is regular. A typical shape changing collision of the three-soliton solution is shown in figure 2 for the parametric choices $k_1 = 0.2+0.3$ i, $k_2 = 0.6+0.4$ i, $k_3 = 0.7+0.2$ i, $\omega_1 = -0.5+0.4$ i, $\omega_2 = -0.7+0.1$ i, $\omega_3 = -0.3 + 0.3$ i, $\alpha_1^{(1)} = 0.5 -$ i, $\alpha_2^{(1)} = 0.5 +$ i, $\alpha_3^{(1)} = 0.3 + 0.2$ i, $\alpha_1^{(2)} = 0.39 + 0.2$ i and $\alpha_2^{(2)} = \alpha_3^{(2)} = 1$. The above three-soliton solution (24) represents the interaction of three solitons and their collision scenario can be well understood by making an asymptotic analysis following the procedure given in section 5 for the two-soliton solution.

We have identified from the asymptotic analysis that for the three-interacting solitons (say, s_1, s_2 and s_3), as in the case of CNLS equations [6, 8], the total transition amplitude of a particular soliton (say s_1) can be expressed as the product of two transition amplitudes

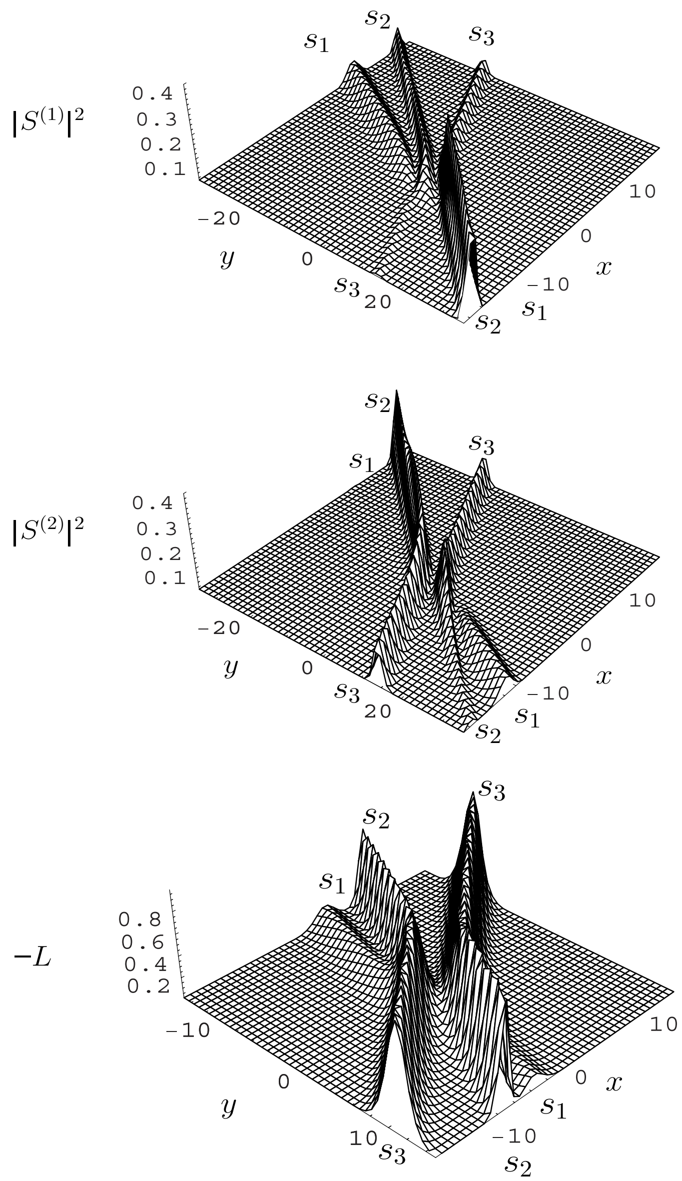

Figure 2. Shape changing collision of the three solitons of the three-wave system, equation (24). The chosen soliton parameters are given in the text. Again note that intensity redistribution occurs only in the $S^{(1)}$ and $S^{(2)}$ components, while elastic collision only occurs in the L component.

which result respectively during the first collision of s_1 with s_2 and during the collision of the outcoming soliton (say s_1') with soliton s_3. In a similar manner the net phase shift acquired by a particular soliton (say s_1) during the complete collision process is equal to the addition of phase shifts experienced by that soliton during its cascaded collisions with s_2 and s_3, respectively. Thus the analysis clearly shows that the multisoliton collision process indeed occurs in a pair-wise manner in the multicomponent (2+1)D LSRI system and there exist no multiparticle effects. The details are similar to the CNLS system [6, 8] and so we do not present them here.

8. Four-soliton (4, 4, 4) solution

Again to obtain the explicit four-soliton solution, we substitute $m = 4$ into the Gram determinant form (7) and obtain an expression involving exponentials. Since it is too lengthy, we do not present the explicit form here. However, we note that the four-soliton solution is characterized by 16 complex parameters, $k_j, \omega_j, \alpha_j^{(1)}, \alpha_j^{(2)}, j = 1, 2, 3, 4$. The nonsingular solution results for the choice $k_{jR}\omega_{jR} < 0, j = 1, 2, 3, 4$. One can check that $k_{jR}\omega_{jR} < 0, j = 1, 2, 3, 4$, are the necessary conditions for the existence of the nonsingular solution and the sufficient condition can be obtained following the procedure mentioned in sections 4 and 7.

8.1. (2, 2, 4)-soliton solution of Ohta et al

In the above-discussed four-soliton solution, we make the choice $\alpha_3^{(1)} = \alpha_4^{(1)} = \alpha_1^{(2)} = \alpha_2^{(2)} = 0$ and also introduce the parametric restrictions,

$$\alpha_1^{(1)} = \frac{(k_1 + k_1^*)(k_1 + k_2^*)(k_1 + k_3^*)(k_1 + k_4^*)}{(k_2 - k_1)(k_3 - k_1)(k_4 - k_1)}, \tag{25a}$$

$$\alpha_2^{(1)} = \frac{(k_2 + k_2^*)(k_2 + k_1^*)(k_2 + k_3^*)(k_2 + k_4^*)}{(k_1 - k_2)(k_3 - k_2)(k_4 - k_2)}, \tag{25b}$$

$$\alpha_3^{(2)} = \frac{(k_3 + k_3^*)(k_3 + k_4^*)(k_3 + k_2^*)(k_3 + k_1^*)}{(k_4 - k_3)(k_2 - k_3)(k_1 - k_3)}, \tag{25c}$$

$$\alpha_4^{(2)} = \frac{(k_4 + k_4^*)(k_4 + k_1^*)(k_4 + k_2^*)(k_4 + k_3^*)}{(k_3 - k_4)(k_2 - k_4)(k_1 - k_4)}, \tag{25d}$$

then one can show that it is exactly equivalent to the (2, 2, 4) soliton expression given by Ohta *et al* [16]. This can be verified by expanding the determinant form with the above parametric restrictions and comparing it with the expanded version of the (2, 2, 4) solution of [16]. The interaction of solitons for the above special case of the four-soliton solution is shown in figure 3 for the choice of parameters $k_1 = 0.5 - 0.2\,\mathrm{i}, k_2 = 0.4 + 0.1\,\mathrm{i}, k_3 = 0.3 - 0.4\,\mathrm{i}, k_4 = 0.4 + 0.6\,\mathrm{i}, \omega_1 = -0.5 + 0.4\,\mathrm{i}, \omega_2 = -0.7 + 0.1\,\mathrm{i}, \omega_3 = -0.3 + 0.3\,\mathrm{i}, \omega_4 = -0.2 + 0.2\,\mathrm{i}$. This is similar to the interaction shown in [16].

9. Soliton solutions of the $(n + 1)$-wave system

We now extend our study to obtain multisoliton solutions of the multicomponent system with arbitrary $(n+1)$ waves, in which we consider n short-wave components and a single long-wave component. The $(n + 1)$-wave system in this case is given by

$$\mathrm{i}\big(S_t^{(j)} + S_y^{(j)}\big) - S_{xx}^{(j)} + LS^{(j)} = 0, \qquad j = 1, 2, \ldots, n, \tag{26a}$$

$$L_t = 2\sum_{j=1}^{n} |S^{(j)}|_x^2. \tag{26b}$$

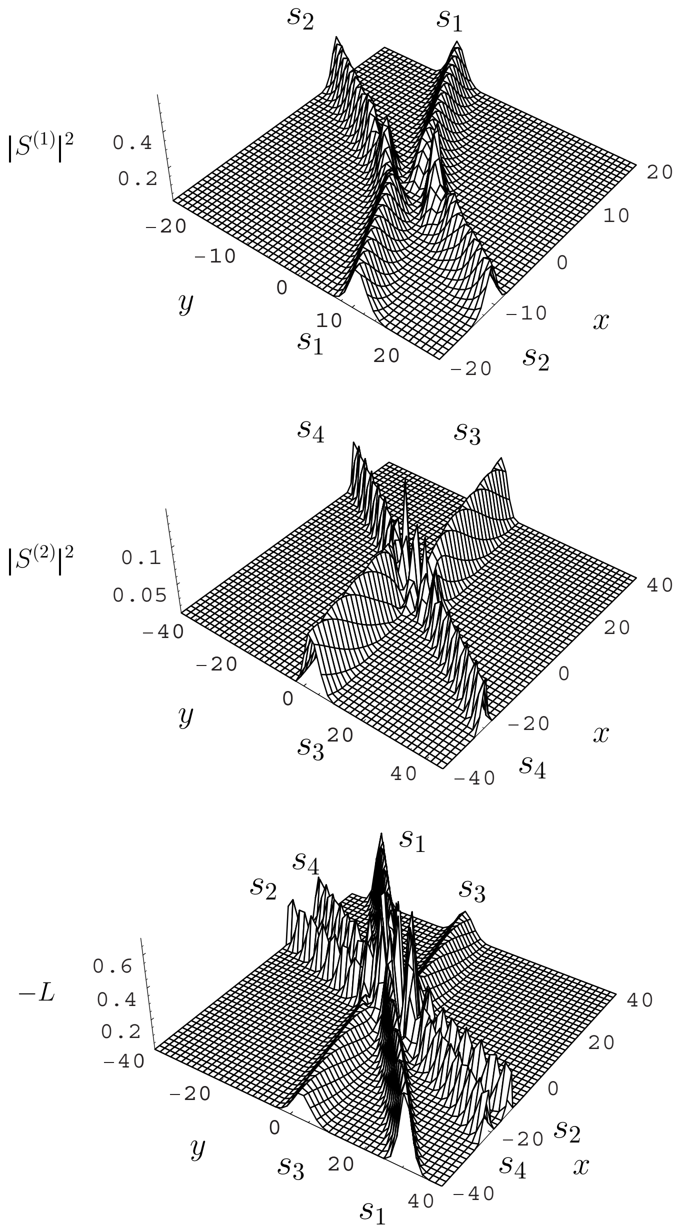

Figure 3. A special case of the four-soliton collision with two solitons in the $S^{(1)}$ and $S^{(2)}$ components and four solitons in the L component, equation (25). Note that this collision scenario is similar to the (2, 2, 4) soliton interaction depicted in figure 2 of [16].

(1) *One-soliton solution*: following the procedure discussed in section 4, we can obtain the one-soliton solution as

$$S^{(j)} = \frac{\alpha_1^{(j)}\, \mathrm{e}^{\eta_1}}{1 + \mathrm{e}^{\eta_1+\eta_1^*+R}}, \qquad j = 1, 2, \ldots n, \tag{27a}$$

$$L = -2\frac{\partial^2}{\partial x^2}\left(\log\left(1 + e^{\eta_1+\eta_1^*+R}\right)\right), \tag{27b}$$

where

$$\eta_1 = k_1 x - \left(ik_1^2 + \omega_1\right)y + \omega_1 t, \qquad e^R = \frac{-\sum_{j=1}^{n}\left(\alpha_1^{(j)}\alpha_1^{(j)*}\right)}{4k_{1R}\omega_{1R}}. \tag{27c}$$

(2) *Two-soliton solution*: a similar procedure results in the two-soliton solution for the multicomponent case with arbitrary $(n + 1)$ waves whose expression can be obtained from equations (14) by just allowing j to run from $1, 2, \ldots, n$ and redefining κ_{il} as $\kappa_{il} = -\sum_{j=1}^{n}\left(\alpha_i^{(j)}\alpha_l^{(j)*}\right)/\left(\omega_i + \omega_l^*\right)$, $i, l = 1, 2$. Now it is straightforward to extend the bilinearization procedure of obtaining one- and two-soliton solutions to obtain multisoliton solutions as in the 1D integrable CNLS equations [8]. Similarly, three- and four-soliton solutions of equation (26) can be obtained by suitably redefining κ_{il} s with $i, l = 1, 2, 3$ and $i, l = 1, 2, 3, 4$, respectively, and fixing the upper limit of the index j, corresponding to the short-wave components as n. The multisoliton solution of the multicomponent case (26) can be written from equation (6) by allowing s to run from 1 to n and redefining the column matrix ψ_j as

$$\psi_j = \left(\alpha_j^{(1)}, \alpha_j^{(2)}, \ldots, \alpha_j^{(n)}\right)^T.$$

Likewise the proof of the multisoliton solution of the multicomponent system also follows the three-wave system discussed in section 3.

10. Conclusion

To conclude, we have obtained explicitly the multi bright plane soliton solutions of the recently reported physically interesting integrable (2+1)-dimensional $(n + 1)$-wave system by applying Hirota's bilinearization procedure. We have also presented the results in a Gram determinant form for the multisoliton solutions of the multicomponent LSRI system along with the necessary proof. We observe that the solitons in the short-wave components can be amplified by merely reducing the pulse width of the long-wave component. The study of collision dynamics shows that the solitons appearing in the short-wave components undergo shape changing collisions with intensity redistribution and amplitude-dependent phase shift. This gives the exciting possibility of soliton collision-based computing in higher dimensional integrable systems also. However, the solitons in the long-wave component always undergo elastic collision.

Acknowledgments

TK acknowledges the support of the Department of Science and Technology, Government of India under the DST Fast Track Project for young scientists. TK and KS thank the Principal and Management of Bishop Heber College, Tiruchirapalli, for constant support and encouragement. The works of MV and ML are supported by a DST-IRPHA project. ML is also supported by a DST Ramanna Fellowship.

References

[1] Kivshar Y S and Agrawal G P 2003 *Optical Solitons: From Fibers to Photonic Crystals* (San Diego: Academic)
[2] Manakov S V 1974 *Sov. Phys. JETP* **38** 248

[3] Ablowitz M J, Prinari B and Trubatch T D 2004 *Discrete and Continuous Nonlinear Schrödinger Systems* (Cambridge: Cambridge University Press)
[4] Radhakrishnan R, Lakshmanan M and Hietarinta J 1997 *Phys. Rev.* E **56** 2213
[5] Kanna T and Lakshmanan M 2001 *Phys. Rev. Lett.* **86** 5043
[6] Kanna T and Lakshmanan M 2003 *Phys. Rev.* E **67** 046617
[7] Vijayajayanthi M, Kanna T and Lakshmanan M 2008 *Phys. Rev.* A **77** 013820
[8] Kanna T, Lakshmanan M, Dinda P T and Akhmediev N 2006 *Phys. Rev.* E **73** 026604
[9] Ablowitz M J, Ohta Y and Trubatch A D 1999 *Phys. Lett.* A **253** 287
[10] Park Q H and Shin H J 2000 *Phys. Rev.* E **61** 3093
[11] Hioe F T 2002 *J. Math. Phys.* **43** 6325
[12] Sukhorukov A A and Akhmediev N N 2003 *Opt. Lett.* **28** 908
[13] Babarro J, Paz-Alonso M J, Michinel H, Salgueiro J R and Olivieri D N 2005 *Phys. Rev.* A **71** 043608
[14] Jakubowski M H, Steiglitz K and Squier R 1998 *Phys. Rev.* E **58** 6752
[15] Steiglitz K 2000 *Phys. Rev.* E **63** 016608
[16] Ohta Y, Maruno K and Oikawa M 2007 *J. Phys. A: Math. Theor.* **40** 7659
[17] Degasperis A, Conforti M, Baronio F and Wabnitz S 2006 *Phys. Rev. Lett.* **97** 093901
[18] Oikawa M, Okamura M and Funakoshi M 1989 *J. Phys. Soc. Japan* **58** 4416
[19] Radha R, Kumar C S, Lakshmanan M, Tang X Y and Lou S Y 2005 *J. Phys. A: Math. Gen.* **38** 9649
[20] Hirota R 1973 *J. Math. Phys.* **14** 805
[21] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)
[22] Maruno K and Ohta Y 2008 *Phys. Lett.* A **372** 4446